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Critical temperature for entanglement transition in Heisenberg Models

Hongchen Fu[†], Allan I Solomon[†] and Xiaoguang Wang[‡]

[†] Quantum Processes Group, The Open University, Milton Keynes, MK7 6AA, U.K.

[‡] Department of Physics, Macquarie University, Sydney, New South Wales 2109, Australia

Abstract. We study thermal entanglement in some low-dimensional Heisenberg models. It is found that in each model there is a critical temperature above which thermal entanglement is absent.

1. Introduction

Entanglement[1] plays an important role in quantum computation and quantum information processing. With appropriate coding, a system of interacting spins, such as described by a Heisenberg hamiltonian, can be used to model a solid-state quantum computer. It is therefore of some significance to study thermal entanglement in Heisenberg models. We find that for each model there is a corresponding critical temperature for transition to the entanglement regime, and the entanglement only occurs below this critical temperature.

2. Measures of entanglement

A pure state described by the wave function $|\Psi\rangle$ is *non-entangled* if it can be factorized as $|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$. Otherwise, it is entangled. A typical example of an entangled state is the Bell state for a bipartite system of two qubits:

$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \quad (1)$$

For such a bipartite system the most popular entanglement measure is the *entanglement of formation*. For a pure state the entanglement of formation is defined as the reduced entropy of either subsystem[2].

For the two-qubit system one can use *concurrence*[3] as a measure of the entanglement. Let ρ_{12} be the density matrix of the pair which may represent either a pure or a mixed state. The concurrence corresponding to the density matrix is defined as

$$C_{12} = \max \{ \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0 \}, \quad (2)$$

where the quantities λ_i are the square roots of the eigenvalues of the operator

$$\varrho_{12} = \rho_{12}(\sigma_1^y \otimes \sigma_2^y) \rho_{12}^* (\sigma_1^y \otimes \sigma_2^y) \quad (3)$$

in descending order. The eigenvalues of ϱ_{12} are real and non-negative even though ϱ_{12} is not necessarily Hermitian. The entanglement of formation is a monotonic function of the concurrence, whose values range from zero, for a non-entangled state, to one, for a maximally entangled state.

3. Heisenberg models

The general N -qubit Heisenberg XYZ model in a magnetic field B is described by the Hamiltonian

$$H = \frac{1}{2} \sum_{n=1}^N (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \sigma_n^z \sigma_{n+1}^z) + \sum_{n=1}^N B_n \sigma_n^z \quad (4)$$

where we assume cyclic boundary conditions $N + 1 \equiv 1$. The Gibbs state of a system in thermodynamic equilibrium is represented by the density operator

$$\rho(T) = \exp(-H/kT)/Z, \quad (5)$$

where $Z = \text{tr}[\exp(-H/kT)]$ is the partition function, k is Boltzmann's constant which we henceforth take equal to 1, and T is the temperature.

As $\rho(T)$ represents a thermal state, the entanglement in the state is called *thermal entanglement*. At $T = 0$, $\rho(0)$ represents the ground state which is pure for the non-degenerate case and mixed for the degenerate case. The ground state may be entangled. At $T = \infty$, $\rho(\infty)$ is a completely random mixture and cannot be entangled.

4. Thermal entanglement in the 2-site Heisenberg model

The density matrix can be obtained[4] as

$$\rho(T) = A \begin{pmatrix} e^{-B/T} & & & \\ & \cosh(J/T) & -\sinh(J/T) & \\ & -\sinh(J/T) & \cosh(J/T) & \\ & & & e^{B/T} \end{pmatrix} \quad (6)$$

where $A = (2 \cosh(J/T) + 2 \cosh(B/T))^{-1}$, and the concurrence

$$C = \max \left\{ \frac{\sinh(J/T) - 1}{\cosh(J/T) + \cosh(B/T)}, 0 \right\}. \quad (7)$$

As the denominator is always positive, the entanglement condition is

$$\sinh(J/T) - 1 > 0 \quad \text{or} \quad T < 1.134J. \quad (8)$$

from which we conclude that

- There is a critical temperature $T_c \sim 1.134J$. The thermal state is entangled when $T < T_c$.
- The critical temperature is independent of the magnetic field B .
- Entanglement occurs only for the antiferromagnetic case ($J > 0$).

5. Thermal entanglement in the 3-site Heisenberg model

We now consider pairwise entanglement in the 3-site Heisenberg model with uniform magnetic field, and also in the presence of a magnetic impurity[5]. The reduced density matrix of two sites can be written as

$$\rho_{12} = \frac{2}{3Z} \begin{pmatrix} u & & & \\ & w & y & \\ & y & w & \\ & & & v \end{pmatrix} \quad (9)$$

The concurrence may be readily obtained as

$$C = \frac{4}{3Z} \max \{ |y| - \sqrt{uv}, 0 \}, \quad (10)$$

In the case of a uniform magnetic field, the entanglement between any two sites is the same due to cyclic symmetry. We therefore need only consider the entanglement between sites 1 and 2. Then

$$\begin{aligned} u(B) &= v(-B) = \frac{3}{2}e^{3\beta B} + \frac{1}{2}e^{\beta B}(2z + z^{-2}) \\ w &= \cosh(\beta B)(2z + z^{-2}) \\ y &= \cosh(\beta B)(z^{-2} - z) \\ Z &= 2 \cosh(3\beta B) + 2 \cosh(\beta B)(2z + z^{-2}). \end{aligned} \quad (11)$$

where $(z = \exp(\beta J))$.

If $B = 0$, one can easily find that the sites 1 and 2 are entangled if and only if

$$2|z^{-2} - z| - 3 - 2z - z^{-2} > 0 \quad (12)$$

from which we conclude that

- There is no entanglement when $J > 0$;
- Entanglement occurs when $J < 0$ and $T < T_c$, where the critical temperature is given by $-1.27J = 1.27|J|$.
- The maximal concurrence is $1/3$, which occurs for $T \rightarrow 0$.

Fig.1 plots the concurrence against τ for different B . From these graphs we see that there exists a critical temperature above which the entanglement vanishes. It is also noteworthy that the critical temperature increases as the magnetic field B increases.

We now consider the case of a single impurity field on the third site; thus $B_1 = B_2 = 0$ and $B_3 = BJ > 0$. In this case the cyclic symmetry is violated and we have to consider the entanglement between sites 1 and 2, and between sites 1 and 3, separately.

Fig.2 plots the concurrence C_{12} and C_{13} against scaled temperature $\tau = kt/|J|$ for different magnetic fields B . From Fig.2(a) we see that when the magnetic field is located at the third site both the antiferromagnetic and ferromagnetic cases are entangled in the range $0 < \tau \leq \tau_c$, where the critical temperature τ_c depends on B . Fig.2(a) also suggests that the concurrence C_{12} tends to 1, namely that the (1,2) entanglement becomes maximal, when $\tau \rightarrow 0$ for large enough B , in both the antiferromagnetic and ferromagnetic cases.

In contrast to the (1,2) case, the entanglement between sites 1 and 3 increases to a maximum with increasing B and then decreases. The lower the τ , the smaller the

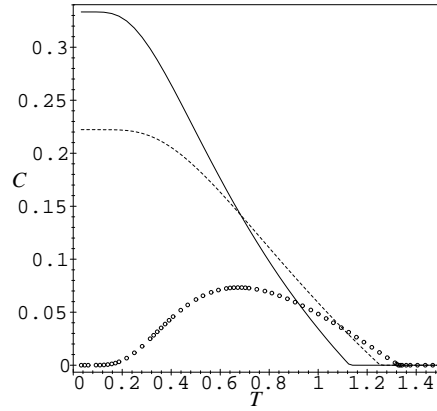


Figure 1. Concurrence as a function of T for different magnetic fields $B = 1$ (solid line), $3/2$ (dashed line), and 2 (circle point line).

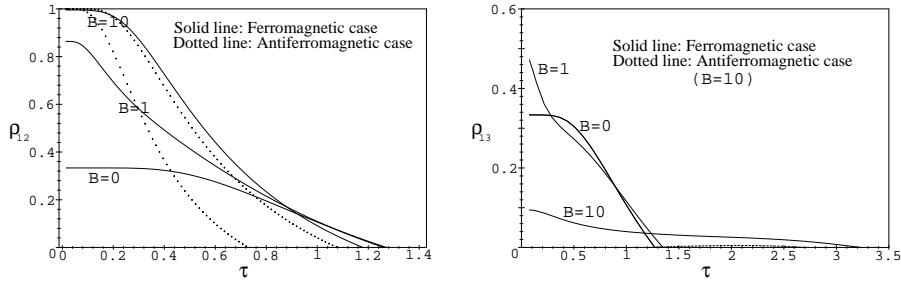


Figure 2. Concurrence C_{12} C_{13} against τ for different B . For antiferromagnetic case (dotted line), $B = 10$.

B at which the concurrence reaches its maximum value. For smaller B , entanglement occurs only in the ferromagnetic case ($J < 0$), while for large enough B (e.g. $B = 10$ in our units), weak entanglement occurs in both the antiferromagnetic and ferromagnetic cases.

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